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Casimir operators of semidirect products of semisimple with Heisenberg groups

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Abstract. Casimir operators for semidirect products of some semisimple groups with Heisenberg groups are computed. The analysis is carried out using dual representations on Fock space, wherein the action of the semidirect products are related to their dual groups, namely certain unitary, orthogonal, and symplectic groups. The compact symplectic group chain is also investigated; by passing to the complexification, groups 'between' the symplectic groups are constructed, which are of the form of semidirect products of symplectic groups with Heisenberg groups.

1. Introduction

If G is a symmetry group, then the universal enveloping algebra U of its Lie algebra $\mathcal{G}$ is an algebra of tensor operators and the invariant operators which form the centralizer of $\mathcal{G}$ in U are called *Casimir invariants*. The importance of these Casimir invariants is due to a fundamental theorem of Chevalley and Racah which states: '*For every semisimple Lie algebra $\mathcal{G}$ of rank n, there exists a set of n algebraically independent generators consisting of Casimir invariants whose eigenvalues determine uniquely the finite-dimensional irreducible representations of $\mathcal{G}$,*'. which in physical terms means that the spectra of the invariant operators associated with G determine the quantum numbers. In this paper we will generalize the Chevalley–Racah theorem for a class of representations of semidirect products of semisimple Lie groups with Heisenberg groups on a Fock space. Quesne [1] has given a set of Casimir operators of a semidirect sum of the unitary, orthogonal, and symplectic algebras with a Heisenberg Weyl algebra. These results are special cases of our results (case $n = N$ in section 2 and $n = [N/2]$ in sections 3 and 4). Moreover, we exhibit a complete set of algebraically independent Casimir operators in every case. Our work makes use of the following set-up.

Let $\mathbb{C}^{n\times N}$ denote the vector space of all $n \times N$ complex matrices. Define a Gaussian measure μ on $\mathbb{C}^{n\times N}$ by

$$d\mu(Z) = \pi^{-nN} \exp[-\operatorname{Tr}(ZZ^*)]dZ \quad Z \in \mathbb{C}^{n\times N} \qquad Z^* = \overline{Z}^{\mathrm{T}} \tag{1.1}$$

where in (1.1) dZ denotes the Lebesque measure on $\mathbb{C}^{n\times N}$. The *Fock space* $\mathcal{F}(\mathbb{C}^{n\times N})$ consists of entire functions on $\mathbb{C}^{n\times N}$ which are square integrable with respect to the Gaussian measure $d\mu(Z)$. Endowed with the inner product

$$(f|g) = \int_{\mathbb{C}^{n\times N}} f(Z)\overline{g(Z)}\, d\mu(Z) \qquad f, g \in \mathcal{F}(\mathbb{C}^{n\times N}) \tag{1.2}$$

$\mathcal{F}(\mathbb{C}^{n\times N})$ has a Hilbert space structure. Actually $\mathcal{F}(\mathbb{C}^{n\times N})$ has a reproducing kernel K, i.e. K is a continuous function from $\mathbb{C}^{n\times N}\times\mathbb{C}^{n\times N}$ to $\mathbb{C}$ such that

$$f(Z)=\int_{\mathbb{C}^{n\times N}} K(Z,Z')f(Z')\,\mathrm{d}\mu(Z')$$

for all $Z\in\mathbb{C}^{n\times N}$ and $f\in\mathcal{F}(\mathbb{C}^{n\times N})$. It can be easily shown that $K(z,z')=\exp\big(\mathrm{Tr}(z(z')^*)\big)$. Moreover, if $\mathcal{F}(\mathbb{C}^{n\times N})$ is endowed with the inner product

$$\langle f,g\rangle = f(D)\overline{g(\overline{Z})}|_{Z=0} \tag{1.3}$$

where $f(D)$ denotes the formal power series obtained by replacing $Z_{\alpha j}$ by the partial derivative $\partial/\partial Z_{\alpha j}$ $(1\leqslant\alpha\leqslant n, 1\leqslant j\leqslant N)$, then the inner products (1.2) and (1.3) coincide on $\mathcal{F}(\mathbb{C}^{n\times N})$. Note that the subspace $\mathcal{P}(\mathbb{C}^{n\times N})$ of all polynomial functions on $\mathbb{C}^{n\times N}$ is dense in $\mathcal{F}(\mathbb{C}^{n\times N})$.

We will make extensive use of the notion of dual representations as defined in Klink and Ton-That [2]. In the context of reductive groups the notion of dual representations coincides with that of complementary pairs of Moshinsky and Quesne [3] and of reductive dual pairs of Howe [4]. Dual representations are defined as follows.

Definition. Let G and G' be two Lie groups (not necessarily reductive). Let R (R') be a representation of G (G') on $\mathcal{F}(\mathbb{C}^{n\times N})$ such that the two actions commute. Assume that R (R') is completely reducible; then the representations R of G and R' of G' are said to be *dual* if the spectral decomposition of R determines that of R' completely.

In the theorems that are proved in this paper we will need this more general notion of dual representations. As a representation space of the joint action $R\otimes R'$ the Hilbert space $\mathcal{F}(\mathbb{C}^{n\times N})$ is decomposed into an orthogonal direct sum

$$\mathcal{F}(\mathbb{C}^{n\times N})=\sum_{(\lambda)}\oplus\mathcal{I}^{(\lambda)}(\mathbb{C}^{n\times N}) \tag{1.4}$$

where in (1.4) the label (λ) characterizes both an equivalence class of an irreducible representation λ_G of G and an equivalence class of an irreducible representation $\lambda_{G'}$ of G', and $\mathcal{I}^{(\lambda)}\equiv\mathcal{I}^{(\lambda)}(\mathbb{C}^{n\times N})$ denotes the (λ)-*isotypic component*, i.e. the direct sum of all irreducible subrepresentations of R (R') that belong to the equivalence class λ_G $(\lambda_{G'})$. Moreover, the restriction of $R\otimes R'$ to $\mathcal{I}^{(\lambda)}$ is irreducible and the sum ranges over all such (λ).

Let $\mathcal{G}$ $(\mathcal{G}')$ denote the Lie algebra of G (G'), let $\mathrm{d}R$ $(\mathrm{d}R')$ denote the differential of R (R'). Then dR $(\mathrm{d}R')$ is a representation of $\mathcal{G}$ $(\mathcal{G}')$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ which we shall refer to as the *infinitesimal action of* R (R'). Let $\mathrm{d}R_{\mathcal{G}}$ $(\mathrm{d}R'_{\mathcal{G}'})$ denote the Lie algebra of operators on $\mathcal{F}(\mathbb{C}^{n\times N})$ generated by the infinitesimal action of R (R'). Let $\mathcal{U}_R\equiv\mathcal{U}(R_G)$ $(\mathcal{U}_{R'}\equiv\mathcal{U}(R'_{G'}))$ be the universal enveloping algebra of $\mathrm{d}R_{\mathcal{G}}$ $(\mathrm{d}R_{\mathcal{G}'})$ then the centre $\mathcal{Z}(\mathcal{U}_R)$ $(\mathcal{Z}(\mathcal{U}_{R'}))$ is called the algebra of all *Casimir invariants of* R (R').

The following fundamental theorem concerning Casimir invariants which is a straightforward generalization of several particular cases proved in [5–7] will be used repeatedly in this paper.

Theorem 1. Let R and R' be two dual representations of G and G', respectively, on the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$. Let $\mathcal{W}_{n\times N}$ denote the Weyl algebra defined by the generators $\{Z_{\alpha i},\ \partial/\partial Z_{\alpha i};\ 1\leqslant\alpha\leqslant n, 1\leqslant i\leqslant N\}$, if $\mathcal{U}_R$ and $\mathcal{U}_{R'}$ are mutual commutants in $\mathcal{W}_{n\times N}$, then the algebras of Casimir invariants of R and R' coincide as an algebra of operators on $\mathcal{F}(\mathbb{C}^{n\times N})$. Moreover, this common algebra is finitely generated.

Our strategy for finding Casimir invariants of representations of semidirect products of groups can then be formulated as follows.

Given a semidirect product of groups (most of the time a semisimple Lie group with a Heisenberg group), say G', find a representation R' of G' on $\mathcal{F}(\mathbb{C}^{n\times N})$ and show that this representation is dual to a representation R of a group G whose R-Casimir invariants are known, then apply theorem 1 to find an explicit set of generators. As we shall see, this strategy works for the representations of semidirect products of groups considered in sections 2, 3 and 4, namely semidirect products of $U(n)$, $Sp(2n,\mathbb{R})$, and $SO^*(2n)$ with Heisenberg groups. However, in section 5 we will exhibit a representation of a semidirect product of a simple Lie group with a Heisenberg group whose dual is a representation of a semidirect product of a semisimple Lie group with an Abelian group, and both algebras of Casimir invariants are equally difficult to determine.

It should be pointed out that in our explicit computation of the generators of Casimir invariants of R, bases (over $\mathbb{R}$) of $\mathcal{G}$ are chosen so that they also constitute bases (over $\mathbb{C}$) of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$, and therefore, we can consider the generators as elements of the centre of the universal enveloping algebra $\mathcal{U}(\mathcal{G}^{\mathbb{C}})$ (with the obvious embedding $\mathcal{U}(\mathcal{G})\subset\mathcal{U}(\mathcal{G}^{\mathbb{C}})$). This principle will be applied throughout this paper.

2. Casimir operators of $U(n)\ltimes_\tau H_n$

Define the joint action $L\otimes R$ of $GL(n,\mathbb{C})\times GL(N,\mathbb{C})$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ by

$$\big[(L\otimes R)(h,g)f\big](Z)=f(h^{-1}Zg)$$

for all $h\in GL(n,\mathbb{C})$, $g\in GL(N,\mathbb{C})$, and $f\in\mathcal{F}(\mathbb{C}^{n\times N})$. Then the representations L and R are dual and we have the following decomposition:

$$\mathcal{F}(\mathbb{C}^{n\times N})=\sum_{(\lambda)}\oplus\mathcal{I}^{(\lambda)}(\mathbb{C}^{n\times N}). \tag{2.1}$$

Here the label (λ) denotes both a signature of an irreducible representation of $GL(n,\mathbb{C})$ of the form $(m_1,\ldots,m_n)$, where $m_1,\ldots,m_n$, are integers which satisfy the condition $m_1\geqslant m_2\geqslant\cdots\geqslant m_n\geqslant 0$, and an irreducible representation of $GL(n,\mathbb{C})$ of the forms

$$\underbrace{(m_1,\ldots,m_n,0,\ldots,0)}_{N}\qquad n\leqslant N$$

$$\underbrace{(m_1,\ldots,m_N,0,\ldots,0)}_{n},\ (m_1,\ldots,m_N)\qquad n>N.$$

The submodule $\mathcal{I}^{(\lambda)}(\mathbb{C}^{n\times N})$ denotes the (λ)-*isotypic component*, i.e. the direct sum of all irreducible submodules of L (R) that belong to the class (λ). The restriction of $L\otimes R$ to $\mathcal{I}^{(\lambda)}(\mathbb{C}^{n\times N})$ is irreducible and the summation ranges over all such (λ). A system of generators of the infinitesimal action of L is given by

$$L_{\alpha\beta}=\sum_{i=1,\ldots,N}Z_{\alpha i}\frac{\partial}{\partial Z_{\beta i}}\qquad 1\leqslant\alpha,\beta\leqslant n. \tag{2.2}$$

Similarly, for R we have

$$R_{ij}=\sum_{\alpha=1,\ldots,n}Z_{\alpha i}\frac{\partial}{\partial Z_{\alpha j}}\qquad 1\leqslant i,j\leqslant N. \tag{2.3}$$

We identify $GL(N-1,\mathbb{C})$ with the subgroup of $GL(N,\mathbb{C})$ consisting of all matrices of the form

$$\left(\begin{array}{c|c}\gamma & 0\\ \hline 0 & 1\end{array}\right)\qquad \gamma\in GL(N-1,\mathbb{C}).$$

If $\widetilde{R}$ denotes the restriction of R to the subgroup $GL(N-1,\mathbb{C})$, then a system of generators of the infinitesimal action of $\widetilde{R}$ is given by

$$\widetilde{R}_{ab}\equiv R_{ab}=\sum_{\alpha=1,\dots,n} Z_{\alpha a}\frac{\partial}{\partial Z_{\alpha b}}\qquad 1\leqslant a,b\leqslant N-1. \tag{2.4}$$

The dual representation of $\widetilde{R}$ is defined as follows. Let ζ denote the column vector

$$\begin{bmatrix} Z_{1N}\\ \vdots \\ Z_{nN}\end{bmatrix}$$

and identify the space of column vectors ζ with $\mathbb{C}^n$, which we equip with the inner product $(\zeta|\zeta')=\zeta'^*\zeta$. Endowed with the bilinear form σ defined by $\sigma(\zeta|\zeta')=-\mathrm{Im}(\zeta|\zeta')$ the vector space $\mathbb{C}^n$ has a symplectic structure. The *Heisenberg group* H_n, which is the set product $\mathbb{C}^n\times\mathbb{R}$ of dimension $2n+1$ over $\mathbb{R}$, is given a group structure by defining the group operation

$$(\zeta,t)\cdot(\zeta',t')=\left(\zeta+\zeta',t+t'+\tfrac{1}{2}\sigma(\zeta|\zeta')\right) \tag{2.5}$$

for all $\zeta,\zeta'\in\mathbb{C}^n$ and $t,t'\in\mathbb{R}$.

Let $\mathcal{F}(\mathbb{C}^n)$ denote the Fock space over the space of column vectors ζ and define a representation π_1 of H_n on $\mathcal{F}(\mathbb{C}^n)$ by the equation

$$[\pi_1(\zeta,t)f](\zeta')=\exp\left[it+\frac{i\sqrt{2}}{2}(\zeta'|\zeta)-|\zeta|^2\right]f\left(\zeta'+i\bar{\zeta}\frac{\sqrt{2}}{2}\right) \tag{2.6}$$

for all $f\in\mathcal{F}(\mathbb{C}^n)$. It is easy to verify that π_1 is an irreducible unitary representation of the Heisenberg group H_n.

The unitary group $U(n)$ acts on H_n by automorphisms via the map $\tau:U(n)\times H_n\to H_n$ defined by $\tau(u,(\zeta,t))=(u\zeta,t)$. The semidirect product $U(n)\ltimes_\tau H(n)$ can be defined by giving the multiplication and inversion operations:

$$\begin{aligned}\left(u_1,(\zeta_1,t_1)\right)\left(u_2,(\zeta_2,t_2)\right)&=\left(u_1u_2,\tau\left((u_2^{-1},(\zeta_1,t_1)\right)\right)\cdot(\zeta_2,t_2)\\ &=\left(u_1u_2,\left(u_2^{-1}\zeta_1+\zeta_2,t_1+t_2+\tfrac{1}{2}\sigma(u_2^{-1}\zeta_1|\zeta_2)\right)\right).\end{aligned}$$

$$\begin{aligned}\left(u,(\zeta,t)\right)^{-1}&=\left(u^{-1},\tau\left(u,(\zeta,t)^{-1}\right)\right)\\ &=\left(u^{-1},(-u\zeta,-t)\right).\end{aligned}$$

Since $\mathcal{F}(\mathbb{C}^{n\times N})$ is isomorphic to $\mathcal{F}(\mathbb{C}^{n\times(N-1)})\otimes\mathcal{F}(\mathbb{C}^{n\times 1})$, we can define the representation $L\ltimes_\tau\pi_1$ of $U(n)\ltimes_\tau H_n$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ as follows.

Writing $Z \in \mathbb{C}^{n\times N}$ as $(Z_1, Z_N) \in \mathbb{C}^{n\times(N-1)} \times \mathbb{C}^{n\times 1}$, and for $\varphi_1 \otimes \varphi_N \in \mathcal{F}(\mathbb{C}^{n\times(N-1)}) \otimes \mathcal{F}(\mathbb{C}^{n\times 1})$, we have

$$\left[\left(L \ltimes_\tau \pi_1\right)\left(u, (\zeta, t)\right)\right](\varphi_1 \otimes \varphi_N)(Z_1, Z_N) = L(u)\varphi_1(Z_1)\pi_1\left(\tau\left(u, (\zeta, t)^{-1}\right)\right)\left(\varphi_N(Z_N)\right) \tag{2.7}$$

where $L(u)\varphi_1(Z_1) = \varphi_1(u^{-1}Z_1)$ and

$$\pi_1\left(\tau\left(u, (\zeta, t)^{-1}\right)\right)\varphi_N(Z_N) = \pi_1\left((-u\zeta, -t)\right)\varphi_N(Z_N)$$

is defined by (2.6). It can easily be shown that $L \ltimes_\tau \pi_1$ is a unitary representation of $U(n) \ltimes_\tau H_n$.

To obtain the infinitesimal action of $L \ltimes_\tau \pi_1$, we first compute the infinitesimal action of π_1 using (2.6). Write $\zeta = (\zeta_1, \ldots, \zeta_n)$ with $\zeta_j = x_j + \mathrm{i}y_j$, $1 \leqslant j \leqslant n$, and identify an infinitesimal generator of the form $((0, \ldots, 0, x_j, 0, \ldots, 0), 0)$ with the real parameter x_j; then an easy computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}x_j}\pi_1(x_j)f(\zeta)\bigg|_{x_j=0} = \frac{\mathrm{i}\sqrt{2}}{2}\left(\zeta_j + \frac{\partial}{\partial\zeta_j}\right)f(\zeta).$$

Similarly, with the infinitesimal generators $((0, \ldots, 0, y_j, 0, \ldots, 0), 0)$ and $((0, \ldots, 0), t)$ we respectively obtain

$$\frac{\mathrm{d}}{\mathrm{d}y_j}\pi_1(y_j)f(\zeta)\bigg|_{y_j=0} = \frac{\sqrt{2}}{2}\left(-\zeta_j + \frac{\partial}{\partial\zeta_j}\right)f(\zeta)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_1(t)f(\zeta)\bigg|_{t=0} = \mathrm{i}f(\zeta).$$

So if the Lie algebra of left invariant vector fields on H_n is spanned by the vector fields P_j, Q_j, $1 \leqslant j \leqslant n$, and R which satisfy the relations

$$[P_j, P_k] = [Q_j, Q_k] = 0 \qquad [P_j, R] = [Q_j, R] = 0$$
$$[P_j, Q_k] = -\delta_{jk}R \qquad 1 \leqslant j, k \leqslant n$$

then we have the representation $d\pi_1$ of the Heisenberg algebra $\mathcal{H}_n$ on $\mathcal{F}(\mathbb{C}^n)$ given by the generators

$$\mathrm{d}\pi_1(P_j) = \frac{\mathrm{i}\sqrt{2}}{2}\left(\zeta_j + \frac{\partial}{\partial\zeta_j}\right) \qquad \mathrm{d}\pi_1(Q_j) = \frac{\sqrt{2}}{2}\left(-\zeta_j + \frac{\partial}{\partial\zeta_j}\right)$$

$1 \leqslant j \leqslant n$ and $\mathrm{d}\pi_1(R) = \mathrm{i}I$, where I is the identity operator on $\mathcal{F}(\mathbb{C}^n)$.

Since

$$\frac{\sqrt{2}}{2}\left(\mathrm{d}\pi_1(Q_j) - \mathrm{i}\mathrm{d}\pi_1(P_j)\right) = \frac{\partial}{\partial\zeta_j}$$

and

$$-\frac{\sqrt{2}}{2}\left(\mathrm{d}\pi_1(Q_j)+\mathrm{id}\pi_1(P_j)\right)=\zeta_j \qquad 1\leqslant j\leqslant n$$

we can use $\zeta_j, \partial/\partial\zeta_j$, $1\leqslant j\leqslant n$, and I as generators of the representation $\mathrm{d}\pi_1$ of $\mathcal{H}_n$. Collecting all the results above we see that the infinitesimal action of $L\ltimes_\tau \pi_1$ induces a representation of the semidirect product (sum) $\mathcal{G}\ell(n,\mathbb{C})\oplus \mathrm{d}\tilde{\tau}\mathcal{H}_n$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ which is generated by the following operators:

$$\begin{array}{llll} L_{\alpha\beta} & 1\leqslant\alpha,\beta\leqslant n & \text{as given by (2.2)} & \\ Z_{\gamma N} & \dfrac{\partial}{\partial Z_{\gamma N}} \quad 1\leqslant\gamma\leqslant n & \text{and I. Moreover} & \qquad (2.8) \end{array}$$

$$[L_{\alpha\beta}, Z_{\gamma N}]=\delta_{\beta\gamma}Z_{\alpha N} \qquad \text{and} \qquad \left[L_{\alpha\beta},\frac{\partial}{\partial Z_{\gamma N}}\right]=-\delta_{\alpha\gamma}\frac{\partial}{\partial Z_{\beta N}}.$$

Equation (2.8) defines the representation of $\mathcal{G}\ell(n,\mathbb{C})\oplus \mathrm{d}\tilde{\tau}\mathcal{H}_n$ which is dual to the representation $\widetilde{R}$ of $\mathcal{G}\ell(N-1,\mathbb{C})$ given by (2.4). We now have the following main result of this section.

Theorem 2. Set $\widetilde{L}_{\mu\nu}=\sum_{i=1,\dots,N-1} Z_{\mu i}\,\partial/\partial Z_{\nu i}$, $1\leqslant\mu,\nu\leqslant n$ and let $[\widetilde{L}]$ denote the $n\times n$ matrix with (μ,ν) entry in $\widetilde{L}_{\mu\nu}$, and for any integer $s>0$ let $\mathrm{Tr}([\widetilde{L}]^s)$ denote the non-commutative trace operator of $[\widetilde{L}]^s$; then the algebra of all Casimir invariant differential operators of the action (2.7) of the semidirect product $U(n)\ltimes_\tau H_n$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ is generated by
(i) the constants and the n algebraically independent Casimir operators $\mathrm{Tr}([\widetilde{L}]^s)$, $1\leqslant s\leqslant n$, if $n<N$;
(ii) the constants and the $N-1$ algebraically independent Casimir operators $\mathrm{Tr}([\widetilde{L}]^s)$, $1\leqslant s\leqslant N-1$ if $n\geqslant N$.

Proof. Clearly we have the isomorphism $\mathcal{F}(\mathbb{C}^{n\times N})=\mathcal{F}(\mathbb{C}^{n\times(N-1)})\otimes\mathcal{F}(\mathbb{C}^{n\times 1})$, so under the restriction to $GL(N-1,\mathbb{C})$ the representation R can be considered as the tensor product representation $\widetilde{R}\otimes I$, where $\widetilde{R}$ is the representation of $GL(N-1,\mathbb{C})$ on $\mathcal{F}(\mathbb{C}^{n\times(N-1)})$ and I is the identity representation of $GL(1,\mathbb{C})$ on $\mathcal{F}(\mathbb{C}^{n\times 1})$. Similar to (2.1) we have the decomposition

$$\mathcal{F}(\mathbb{C}^{n\times(N-1)})=\sum_{(\mu)}\oplus\mathcal{I}^{(\mu)}(\mathbb{C}^{n\times(N-1)}) \qquad (2.9)$$

where (μ) denotes both the signature of an irreducible representation of $GL(n,\mathbb{C})$ and an irreducible representation of $GL(N-1,\mathbb{C})$. Thus
(i) If $n<N$ then (μ) is of the form $(\ell_1,\dots,\ell_n)$, with $\ell_1\geqslant\ell_2\geqslant\dots\geqslant\ell_n\geqslant 0$ for $GL(n,\mathbb{C})$, and of the form

$$\underbrace{(\ell_1,\dots,\ell_n,0,\dots,0)}_{N-1}$$

for $GL(N-1,\mathbb{C})$.

(ii) If $n \geqslant N$ then (μ) is of the form

$$\underbrace{(\ell_1, \ldots, \ell_{N-1}, 0, \ldots, 0)}_{n}$$

for $GL(n, \mathbb{C})$ and of the form $(\ell_1, \ldots, \ell_{N-1})$ for $GL(N-1, \mathbb{C})$.

In this context the dual representation of $\widetilde{R}$ is the representation $\widetilde{L}$ of $GL(n, \mathbb{C})$ on $\mathcal{F}(\mathbb{C}^{n\times(N-1)})$ defined by

$$\widetilde{L}(h)\varphi_1(Z_1) = \varphi_1(h^{-1}Z_1) \qquad \forall\, h \in GL(n, \mathbb{C}) \quad \text{and} \quad \forall\, \varphi_1 \in \mathcal{F}(\mathbb{C}^{n\times(N-1)}). \tag{2.10}$$

The infinitesimal generators of $\widetilde{L}$ are

$$\widetilde{L}_{\mu\nu} = \sum_{i=1,\ldots,N-1} Z_{\mu i} \frac{\partial}{\partial Z_{\nu i}} \qquad 1 \leqslant \mu, \nu \leqslant n. \tag{2.11}$$

If $[\widetilde{R}]$ denotes the $(N-1)\times(N-1)$ matrix whose (a, b) entry, $1 \leqslant a, b \leqslant N-1$, is $\widetilde{R}_{ab}$ as defined by (2.4), then it follows from lemmas 3.1 and 3.2 of Klink and Ton-That [5] that if $n < N$ then the algebra of all Casimir invariants of the representation $\widetilde{R}$ of $GL(N-1, \mathbb{C})$ is generated by the constants and by the n algebraically independent Casimir operators $\mathrm{Tr}([\widetilde{R}])$, $\mathrm{Tr}([\widetilde{R}]^2), \ldots, \mathrm{Tr}([\widetilde{R}]^n)$. Similarly the algebra of all Casimir invariants of the dual representation (to $\widetilde{R}$) $\widetilde{L}$ of $GL(n, \mathbb{C})$ is generated by the constants and by the n algebraically independent Casimir operators $\mathrm{Tr}([\widetilde{L}])$, $\mathrm{Tr}([\widetilde{L}]^2), \ldots, \mathrm{Tr}([\widetilde{L}]^n)$. Moreover these two algebras of Casimir invariants coincide. The same conclusion holds for the case $n \geqslant N$ except we now have $N-1$ algebraically independent Casimir operators.

We have shown that the representation $L \ltimes_\tau \pi_1$ of $U(n) \ltimes_\tau H_n$ is dual to the representation $\widetilde{R}$ of $U(N-1)$ on $\mathcal{F}(\mathbb{C}^{n\times N})$. So by theorem 1 the algebra of all Casimir invariants of the representation $L \ltimes_\tau \pi_1$ of $U(n) \ltimes_\tau H_n$ coincides with the algebra of Casimir invariants of the representation $\widetilde{R}$ of $U(N-1)$ which, in turn, coincides with the algebra of Casimir invariants of the representation $\widetilde{L}$ of $U(n)$, and hence the conclusion of the theorem follows. □

3. Casimir operators of $\widetilde{Sp}(2n, \mathbb{R}) \ltimes_\tau H_n$

We consider next the restriction of the representation R on $\mathcal{F}(\mathbb{C}^{n\times N})$ to the orthogonal group of order N. As mentioned in the introduction, for the purpose of finding Casimir operators it suffices to compute the infinitesimal and its dual actions, but for the sake of completeness we will also describe briefly the dual group actions. We realize the orthogonal group of order N as follows. Let S denote an $N \times N$ matrix such that $S = S^{-1} = S^{\mathrm{T}}$ and let $G^{\mathbb{C}} = \{g \in GL(N, \mathbb{C}) : gSg^{\mathrm{T}} = S\}$; let G denote the compact real form of $G^{\mathbb{C}}$. Then the infinitesimal generators of the restriction of R to G are

$$R^G_{ij} = \sum_{\substack{\alpha=1,\ldots,n\\ k=1,\ldots,N}} (Z_{\alpha i}S_{kj} - Z_{\alpha j}S_{ki}) \frac{\partial}{\partial Z_{\alpha k}} \qquad 1 \leqslant i, j \leqslant N. \tag{3.1}$$

For example, when $S = I_N$ we have the standard form of the orthogonal groups $O(N)$ and

$$R^G_{ij} = \sum_{\alpha=1,\ldots,n} \left(Z_{\alpha i} \frac{\partial}{\partial Z_{\alpha j}} - Z_{\alpha j} \frac{\partial}{\partial Z_{\alpha i}} \right) \qquad 1 \leqslant i, j \leqslant N.$$

To find the dual representation of this infinitesimal action, we consider the polynomials $p_{\alpha\beta}$ defined by

$$p_{\alpha\beta}(z) = (ZsZ^{\mathrm{T}})_{\alpha\beta} \qquad 1 \leqslant \alpha, \beta \leqslant n \tag{3.2}$$

where $(\ )_{\alpha\beta}$ denotes the (α, β) entry of the matrix $(\ )$. Since $gSg^{\mathrm{T}} = S$ for all $g \in G^{\mathbb{C}}$, we have

$$\begin{aligned} R^G_{(g)} p_{\alpha\beta}(Z) = p_{\alpha\beta}(Zg) &= \big((Zg)S(Zg)^{\mathrm{T}}\big)_{\alpha\beta} \\ &= (ZgSg^{\mathrm{T}}Z^{\mathrm{T}})_{\alpha\beta} \\ &= (ZSZ^{\mathrm{T}})_{\alpha\beta} = p_{\alpha\beta}(Z). \end{aligned}$$

Thus the $p_{\alpha\beta}$'s are R^G-invariant; in fact, by the theory of polynomial invariants they are algebraically independent and together with the constants they generate all R^G-invariant polynomials [8]. Define

$$(D_{\alpha\beta} f)(Z) = p_{\alpha\beta}(D) f(Z) = \left(\sum_{i,j=1,\dots,N} \frac{\partial}{\partial Z_{\alpha j}} S_{ij} \frac{\partial}{\partial Z_{\beta i}} \right) f(Z)$$

for all $f \in \mathcal{F}(\mathbb{C}^{n\times N})$ and all $\alpha, \beta = 1, \dots, n$, and recall that

$$L_{\alpha\beta} = \sum_{i=1}^{N} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}}.$$

Then an easy computation shows that

$$\begin{aligned} [D_{\alpha\beta}, p_{\mu\nu}] &= \delta_{\alpha\mu}\big(L_{\nu\beta} + \tfrac{1}{2} N\delta_{\nu\beta}\big) + \delta_{\alpha\nu}\big(L_{\mu\beta} + \tfrac{1}{2} N\delta_{\mu\beta}\big) + \delta_{\beta\mu}\big(L_{\nu\alpha} + \tfrac{1}{2} N\delta_{\nu\alpha}\big) \\ &\quad + \delta_{\beta\nu}\big(L_{\mu\alpha} + \tfrac{1}{2} N\delta_{\mu\alpha}\big) \\ [L_{\alpha\beta}, p_{\mu\nu}] &= \delta_{\beta\mu} p_{\alpha\nu} + \delta_{\beta\nu} p_{\alpha\mu} \\ [L_{\alpha\beta}, D_{\mu\nu}] &= -\delta_{\alpha\mu} D_{\beta\nu} - \delta_{\alpha\nu} D_{\beta\mu} \end{aligned}$$

Set $P_{\alpha\beta} = -p_{\alpha\beta}$, $E_{\alpha\beta} = L_{\alpha\beta} + 1/2\, N\delta_{\alpha\beta}$ and if $T^{\dagger}$ denotes the adjoint of the operator T, then it follows immediately that for all $\alpha, \beta, \mu, \nu = 1, \dots, n$

$$\begin{aligned} &[E_{\alpha\beta}, E_{\mu\nu}] = \delta_{\beta\mu} E_{\alpha\nu} - \delta_{\alpha\nu} E_{\mu\beta} \\ &[E_{\alpha\beta}, P_{\mu\nu}] = \delta_{\beta\mu} P_{\alpha\nu} + \delta_{\beta\nu} P_{\alpha\mu} \\ &[E_{\alpha\beta}, D_{\mu\nu}] = -\delta_{\alpha\mu} D_{\beta\nu} - \delta_{\alpha\nu} D_{\beta\mu} \\ &[P_{\alpha\beta}, D_{\mu\nu}] = \delta_{\alpha\mu} E_{\nu\beta} + \delta_{\alpha\nu} E_{\mu\beta} + \delta_{\beta\mu} E_{\nu\alpha} + \delta_{\beta\nu} E_{\mu\alpha} \\ &[P_{\alpha\beta}, P_{\mu\nu}] = [D_{\alpha\beta}, D_{\mu\nu}] = 0 \\ &P_{\alpha\beta} = P_{\beta\alpha} \qquad D_{\alpha\beta} = D_{\beta\alpha} \\ &P^{\dagger}_{\alpha\beta} = D_{\alpha\beta} \qquad D^{\dagger}_{\alpha\beta} = P_{\alpha\beta} \qquad E^{\dagger}_{\alpha\beta} = E_{\beta\alpha}. \end{aligned} \tag{3.3}$$

If

$$\sigma_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

then the real symplectic group $Sp(2n, \mathbb{R})$ can be realized as the set of all matrices $h \in GL(2n, \mathbb{R})$ such that $h\sigma_n h^{\mathrm{T}} = \sigma_n$ and its Lie algebra $sp(2n, \mathbb{R})$ consists of all matrices of the form

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_1^{\mathrm{T}} \end{pmatrix}$$

where X_1, X_2, and X_3 are real $n \times n$ matrices with X_2 and X_3 symmetric. If M_{ij} denotes the $2n \times 2n$ matrix with the (i, j) entry equal to 1 and all the other entries equal to 0, then the set

$$\{M_{\alpha\beta} - M_{\beta+n,\alpha+n}, M_{\alpha,\beta+n} + M_{\beta,\alpha+n}, M_{\alpha+n,\beta} + M_{\beta+n,\alpha}\} \qquad 1 \leqslant \alpha, \beta \leqslant n$$

forms a basis of $Sp(2n, \mathbb{R})$ and the linear map which sends $M_{\alpha\beta} - M_{\beta+n,\alpha+n}$ to $E_{\alpha\beta}$, $M_{\alpha,\beta+n} + M_{\beta,\alpha+n}$ to $P_{\alpha\beta}$, and $M_{\alpha+n,\beta} + M_{\beta+n,\alpha}$ to $D_{\alpha\beta}$ defines a faithful representation of $Sp(2n, \mathbb{R})$ on $\mathcal{F}(\mathbb{C}^{n\times N})$. By construction this representation is dual to the infinitesimal action of R^G. Let G' denote $Sp(2n, \mathbb{R})$ and let $L^{G'}$ denote the dual representation of R^G, then the pair $(R^G, L^{G'})$ forms the oscillator representation of the pair (G, G') on $\mathcal{F}(\mathbb{C}^{n\times N})$ [4]. Actually, to be precise, $L^{G'}$ is a unitary representation of the two-sheeted covering group, G'_2 called the metaplectic group, of G'. This representation is explicitly given in Kashiwara and Vergne [9, p 11] as a representation of G'_2 on the Schrödinger space $L^2(\mathbb{R}^{n\times N})$, and to obtain the representation $L^{G'_2}$ on the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$ we must use the unitary Bargmann–Segal transform from the Schrödinger space onto the Fock space [10, 11]; the final form of this representation is quite complicated and, since we do not need it, we will not exhibit it here.

As in section 2 we have the following decomposition:

$$\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(\lambda_G)} \oplus \mathcal{I}^{(\lambda_G)}(\mathbb{C}^{n\times N}) \tag{3.4}$$

where the label (λ_G) denotes both a signature of an irreducible representation of G and a non-singular Harish–Chandra parameter of a discrete series π_λ of G' [12]. More precisely, to each (λ_G) there corresponds uniquely a sequence of integers $m_1, m_2, \ldots$ which satisfy the (dominant condition $m_1 \geqslant m_2 \geqslant \cdots \geqslant 0$ and form the signature $(m_1, \ldots, m_n)$ if $n < [N/2]$ and the signature $(m_1, \ldots, m_{[N/2]})$ if $[N/2] \leqslant n$, where $[N/2]$ denotes the integral part of $N/2$.

Let $[R^G]$ denote the matrix whose (i, j) entry is given by (3.1), then the algebra of Casimir operators of the representation R^G is generated by the algebraically independent non-commutative differential operators $\mathrm{Tr}([R^G]^{2i})$; $1 \leqslant i \leqslant n$ if $n < [N/2]$, and $1 \leqslant i \leqslant [N/2]$ if $[N/2] \leqslant n$. (See Barut and Raczka [13] and Želebenko [14] for details.) In fact, letting

$$S = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$$

we obtain the operators of the form given in Želebenko [14], and to apply theorem 1, we should use the form

$$S = \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}$$

if $N = 2p$ and

$$S = \begin{pmatrix} 0 & 0 & 1_p \\ 0 & 1 & 0 \\ 1_p & 0 & 0 \end{pmatrix}$$

if $N = 2p + 1$.

Let E, P, D denote the matrices whose (α, β) entry, $1 \leqslant \alpha, \beta \leqslant n$, is $E_{\alpha\beta}$, $P_{\alpha\beta}$, and $D_{\alpha\beta}$, respectively, as given in (3.3) and set

$$[L^{G'}] = \begin{bmatrix} E & P \\ D & -E^{\mathrm{T}} \end{bmatrix}$$

then according to theorem 1 the algebra of Casimir operators of the representation $L^{G'}$ coincides with the algebra of Casimir operators of the representation R^G. Furthermore, it is generated by the algebraically independent operators $\mathrm{Tr}([L^{G'}]^{2i})$; $1 \leqslant i \leqslant n$ if $n < [N/2]$, and $1 \leqslant i \leqslant n$ if $[N/2] \leqslant n$. Note that the $E_{\alpha\beta}$ generate a subalgebra $\mathcal{T}_{\mathbb{C}}$, the $P_{\alpha\beta}$ generate a subalgebra $\mathcal{P}_+$, and the $D_{\alpha\beta}$ generate a subalgebra $\mathfrak{P}_-$ in such a way that $\mathcal{T}_{\mathbb{C}} + \mathfrak{P}_+ + \mathfrak{P}_-$ gives a realization of the Lie algebra $sp(2n, \mathbb{C})$.

Set $G \equiv G_N$ and identify G_{N-1} with the subgroup of G_N which consists of all matrices of the form

$$\left(\begin{array}{c|c} \gamma & 0 \\ \hline 0 & 1 \end{array}\right) \qquad \gamma \in G_{N-1}.$$

If $\widetilde{R}^G$ denotes the restriction of R^G to G_{N-1}, then a system of generators of the infinitesimal action of $\widetilde{R}^G$ is given by

$$\widetilde{R}^G_{ab} = \sum_{\substack{\alpha=1,\ldots,n \\ k=1,\ldots,N-1}} \left(Z_{\alpha a} S^{N-1}_{kb} - Z_{\alpha b} S^{N-1}_{ka}\right) \frac{\partial}{\partial Z_{\alpha k}} \qquad 1 \leqslant a, b \leqslant N-1 \tag{3.5}$$

where S^{N-1} denotes the matrix of a non-degenerate symmetric bilinear form of $\mathbb{C}^{N-1}$. We will only give the infinitesimal dual action of $\widetilde{R}^G$, the dual group action can be obtained in a similar fashion as in the case $\widetilde{R}$ of section 2. This infinitesimal dual action is a representation of the semidirect product (sum) of $sp(2n, \mathbb{C}) \oplus_{\mathrm{d}\tau} \mathcal{H}_n$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ which is generated by the following operators

$$E_{\alpha\beta} \quad P_{\alpha\beta} \quad \text{and} \quad D_{\alpha\beta} \qquad 1 \leqslant \alpha, \beta \leqslant n \ \text{ as given by (3.3)}$$

$$Z_{\gamma N} \quad \frac{\partial}{\partial Z_{\gamma N}} \qquad 1 \leqslant \gamma \leqslant n \ \text{ and I. Moreover, we have}$$

$$[E_{\alpha\beta}, Z_{\gamma N}] = \delta_{\beta\gamma} Z_{\alpha N} \qquad \left[E_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma N}}\right] = -\delta_{\alpha\gamma} \frac{\partial}{\partial Z_{\beta N}} \tag{3.6}$$

$$[P_{\alpha\beta}, Z_{\gamma N}] = 0 \qquad [D_{\alpha\beta}, Z_{\gamma N}] = \delta_{\beta\gamma} \frac{\partial}{\partial Z_{\alpha N}} + \delta_{\alpha\gamma} \frac{\partial}{\partial Z_{\beta N}}$$

$$\left[P_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma N}}\right] = -\delta_{\alpha\nu} Z_{\beta N} - \delta_{\beta\gamma} Z_{\alpha N} \qquad \left[D_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma N}}\right] = 0$$

Let

$$\widetilde{P}_{\alpha\beta} = -\sum_{i,j=1,\dots,N-1} Z_{\alpha j} S_{ij}^{N-1} Z_{\beta i}$$

$$\widetilde{D}_{\alpha\beta} = \sum_{i,j=1,\dots,N-1} \frac{\partial}{\partial Z_{\alpha j}} S_{ij}^{N-1} \frac{\partial}{\partial Z_{\beta i}} \qquad 1 \leqslant \alpha, \beta \leqslant n \tag{3.7}$$

$$\widetilde{E}_{\alpha\beta} = \sum_{i=1,\dots,N-1} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}} + \frac{1}{2}(N-1)\delta_{\alpha\beta}.$$

Let $\widetilde{E}$, $\widetilde{P}$, $\widetilde{D}$ denote the matrices whose (α, β) entry, $1 \leqslant \alpha, \beta \leqslant n$, is $\widetilde{E}_{\alpha\beta}$, $\widetilde{P}_{\alpha\beta}$, and $\widetilde{D}_{\alpha\beta}$, respectively, as given by (3.7), and set

$$[\widetilde{L}^{G'}] = \begin{bmatrix} \widetilde{E} & \widetilde{P} \\ \widetilde{D} & -\widetilde{E}^{\mathrm{T}} \end{bmatrix}$$

then we have the following theorem.

Theorem 3. Let $\widetilde{S}p(2n, \mathbb{R})$ denote the two-sheeted covering group of the symplectic group $Sp(2n, \mathbb{R})$ (the metaplectic group) then the algebra of all Casimir invariant differential operators of the representation of the semidirect product $\widetilde{S}p(2n, \mathbb{R}) \ltimes_\tau H_n$ which is dual to the representation $\widetilde{R}^G$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ is generated by:
(i) the constants and the n algebraically independent Casimir operators $\mathrm{Tr}([\widetilde{L}^{G'}]^{2s})$, $1 \leqslant s \leqslant n$, if $n < [N/2]$;
(ii) the constants and the $[N/2]$ algebraically independent Casimir operators $\mathrm{Tr}([\widetilde{L}^{G'}]^{2s})$, $1 \leqslant s \leqslant [N-1]/2$, if $[N/2] \leqslant n$.

Proof. We have the decompositions

$$\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(\mu_{G_{N-1}})} \oplus \big(\mathcal{I}^{(\mu_{G_{N-1}})}(\mathbb{C}^{n\times(N-1)}) \otimes \mathcal{F}(\mathbb{C}^{n\times 1})\big) \tag{3.8a}$$

$$\mathcal{F}(\mathbb{C}^{n\times(N-1)}) = \sum_{(\mu_{G_{N-1}})} \oplus \mathcal{I}^{(\mu_{G_{N-1}})}(\mathbb{C}^{n\times(N-1)}) \tag{3.8b}$$

where in (3.8*a*) the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$ is decomposed into a direct sum of isotypic components under the dual actions of $G_{N-1} \times I$ and $\widetilde{S}p(2n, \mathbb{R}) \ltimes_\tau H_n$, and in (3.8*b*) the Fock space $\mathcal{F}(\mathbb{C}^{n\times(N-1)})$ is decomposed into isotypic components of the dual actions of G_{N-1} and $G'_{N-1} \approx \widetilde{S}p(2n, \mathbb{R})$. Thus if:
(i) $n < [N/2]$ then $n \leqslant [N/2-1]$ and $(\mu_{G_{N-1}})$ corresponds to the signature $(\ell_1, \dots, \ell_n)$, $\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant 0$.
(ii) $n \geqslant [N/2]$ then $n \geqslant [(N-1)/2]$ and $(\mu_{G_{N-1}})$ corresponds to the signature $(\ell_1, \dots, \ell_{[(N-1)/2]})$.

It follows from [7] that the algebra of all Casimir invariants of the dual representations $\widetilde{R}^G$ and $\widetilde{L}^{G'}$ coincide and they are generated either by the system $\{\mathrm{Tr}([\widetilde{R}^G]^s)\}_s$, where $[R^G]$ is the $(N-1) \times (N-1)$ matrix whose (a, b) entry is $\widetilde{R}^G_{ab}$, is given by (3.5), or by the system $\{\mathrm{Tr}([\widetilde{L}^{G'}]^s)\}_s$, with $1 \leqslant s \leqslant n$ if $n < [N/2]$, and $1 \leqslant s \leqslant [(N-1)/2]$ if $[N/2] \leqslant n$. But since the actions of $G_{N-1} \times I$ and $\widetilde{S}p(2n, \mathbb{R}) \ltimes_\tau H_n$ on $\mathcal{F}(\mathbb{C}^{n\times(N-1)}) \otimes \mathcal{F}(\mathbb{C}^{n\times 1})$ are dual, theorem 1 implies that the algebra of Casimir invariants of the action of $\widetilde{S}p(2n, \mathbb{R}) \ltimes_\tau H_n$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ coincide with that of $\widetilde{R}^G \otimes I$, and hence, by transitivity the conclusion of the theorem follows. □

4. Casimir operators of $SO^*(2n) \ltimes_\tau (H_n \times H_n)$

Let N be an *even integer* and consider the restriction of the representation R on the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$ to the symplectic group of order N. The symplectic group, denoted by G in this section (not to be confused with $\bar{G}$ in sections 2 and 3) is the subgroup of $GL(N, \mathbb{C})$ that preserves a non-degenerate skew symmetric bilinear form of $\mathbb{C}^N$. Let σ be the matrix of this form with respect to some basis of $\mathbb{C}^N$, then $\sigma^{-1} = -\sigma = \sigma^{\mathrm{T}}$ and

$$G = \{g \in GL(N, \mathbb{C}) | g\sigma g^{\mathrm{T}} = \sigma\}.$$

Let R^G denote the restriction of R to G, then a system of generators of the infinitesimal action of R^G is given by

$$R^G_{ij} = \sum_{\substack{\alpha=1,\ldots,n \\ k=1,\ldots,N}} \left(Z_{\alpha i}\sigma_{kj} + Z_{\alpha j}\sigma_{ki}\right) \frac{\partial}{\partial Z_{\alpha k}} \qquad 1 \leqslant i, j \leqslant N. \tag{4.1}$$

As in the case of the orthogonal group, to find the dual representation of the infinitesimal action of R^G we consider the non-constant generators of all R^G-invariant polynomials defined by

$$p_{\alpha\beta}(Z) = (Z\sigma Z^{\mathrm{T}})_{\alpha\beta} = \sum_{i,j=1,\ldots,N} Z_{\alpha j}\sigma_{ji} Z_{\beta i} \qquad 1 \leqslant \alpha, \beta \leqslant n$$

and the G-invariant differential operators

$$p_{\alpha\beta}(D) = \sum_{i,j=1,\ldots,N} \frac{\partial}{\partial Z_{\alpha j}} \sigma_{ji} \frac{\partial}{\partial Z_{\beta i}} \qquad 1 \leqslant \alpha, \beta \leqslant n.$$

Set $P_{\alpha\beta} = -p_{\alpha\beta}$, $D_{\alpha\beta} = p_{\alpha\beta}(D)$, and $E_{\alpha\beta} = L_{\alpha\beta} + [N/2]\delta_{\alpha\beta}$, then an easy computation shows that for all $\alpha, \beta, \mu, \nu = 1, \ldots, n$ we have

$$\begin{aligned}
&[E_{\alpha\beta}, E_{\mu\nu}] = \delta_{\beta\mu} E_{\alpha\nu} - \delta_{\alpha\nu} E_{\mu\beta} \\
&[E_{\alpha\beta}, P_{\mu\nu}] = \delta_{\beta\mu} P_{\alpha\nu} + \delta_{\beta\nu} P_{\mu\alpha} \\
&[E_{\alpha\beta}, D_{\mu\nu}] = -\delta_{\alpha\mu} D_{\beta\nu} - \delta_{\alpha\nu} D_{\mu\beta} \\
&[P_{\alpha\beta}, D_{\mu\nu}] = \delta_{\alpha\mu} E_{\nu\beta} + \delta_{\beta\nu} E_{\mu\alpha} - \delta_{\alpha\nu} E_{\mu\beta} - \delta_{\beta\mu} E_{\nu\alpha} \\
&[P_{\alpha\beta}, P_{\mu\nu}] = [D_{\alpha\beta}, D_{\mu\nu}] = 0 \\
&P_{\alpha\beta} = -P_{\beta\alpha} \qquad D_{\alpha\beta} = -D_{\beta\alpha} \\
&P^\dagger_{\alpha\beta} = D_{\alpha\beta} \qquad D^\dagger_{\alpha\beta} = P_{\alpha\beta} \qquad E^\dagger_{\alpha\beta} = E_{\beta\alpha}.
\end{aligned} \tag{4.2}$$

If

$$J_n = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} \qquad \text{and} \qquad S_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$$

then

$$SU(n,n) = \{g \in GL(2n,\mathbb{C}) | g J_n g^\dagger = J_n\}$$

and

$$SO^*(2n) = \{g \in SU(n,n) | g S_n g^{\mathrm{T}} = S_n\}$$

and the Lie algebra $SO^*(2n)$ of $SO^*(2n)$ consists of all matrices of the form

$$\begin{pmatrix} A & B \\ -\bar{B} & -A^{\mathrm{T}} \end{pmatrix}$$

where A, B are $n \times n$ matrices with A skew-Hermitian and B skew-symmetric. The complexification of $SO^*(2n)$, which can be identified with $SO(2n,\mathbb{C})$, is the set of all matrices of the form

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^{\mathrm{T}} \end{pmatrix}$$

where X_1, X_2, and X_3 are complex matrices of order n, and X_2, X_3 are skew-symmetric. The set

$$\{M_{\alpha\beta} - M_{\beta+n,\alpha+n}, M_{\alpha,\beta+n} - M_{\beta,\alpha+n}, M_{\beta+n,\alpha} - M_{\alpha+n,\beta}\} \qquad 1 \leqslant \alpha, \beta \leqslant n$$

forms a basis of $SO(2n,\mathbb{C})$ and the linear map which sends $M_{\alpha\beta} - M_{\beta+n,\alpha+n}$ to $E_{\alpha\beta}$, $M_{\alpha,\beta+n} - M_{\beta,\alpha+n}$ to $P_{\alpha\beta}$, and $M_{\beta+n,\alpha} - M_{\alpha+n,\beta}$ to $D_{\alpha,\beta}$ defines a faithful representation of $SO^*(2n)$ on $\mathcal{F}(\mathbb{C}^{n\times N})$. By construction this representation is dual to the infinitesimal action of R^G. Let G' denote $SO^*(2n)$ and let $L^{G'}$ denote the dual representation of R^G, then the pair $(R^G, L^{G'})$ forms the oscillator representation of the pair (G, G') on $\mathcal{F}(\mathbb{C}^{n\times N})$ (cf [4]). As in section 3 we refer the reader to [9–11] for the global action of $L^{G'}$ on $\mathcal{F}(\mathbb{C}^{n\times N})$.

Again we have the decomposition

$$\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(\lambda_G)} \oplus \mathcal{I}^{(\lambda_G)}(\mathbb{C}^{n\times N}) \tag{4.3}$$

where (λ_G) corresponds to the signature $(m_1, \ldots, m_n)$ for G' and the signature

$$\underbrace{(m_1, \ldots, m_n, 0, \ldots, 0)}_{N/2}$$

for G if $n < N/2$, and

$$\underbrace{(m_1, \ldots, m_{N/2}, 0, \ldots, 0)}_{n}$$

for G' and $(m_1, \ldots, m_{N/2})$ for G if $n \geqslant N/2$.

Let $[R^G]$ denote the matrix whose (i, j) entry is given by (4.1), let

$$[L^{G'}] = \begin{bmatrix} E & P \\ D & -E^{\mathrm{T}} \end{bmatrix}$$

denote the $2n \times 2n$ matrix whose entries $E_{\alpha\beta}$, $P_{\alpha,\beta}$, and $D_{\alpha,\beta}$ are given by (4.2). Then by theorem 1 the algebras of the representations R^G and $L^{G'}$ coincide and they are generated by the constants and by the algebraically independent non-commutative trace operators $\mathrm{Tr}([R^G]^{2i})$, or equivalently, $\mathrm{Tr}([L^{G'}]^{2i})$, $1 \leqslant i \leqslant n$ if $n < N/2$, and $1 \leqslant i \leqslant N/2$ if $n \geqslant N/2$. Again we remark that the $E_{\alpha\beta}$ generate a subalgebra $\mathcal{T}_{\mathbb{C}}$, the $P_{\alpha\beta}$ generate a subalgebra $\mathfrak{P}_+$, and the $D_{\alpha\beta}$ generate a subalgebra $\mathfrak{P}_-$ in such a way that $\mathcal{T}_{\mathbb{C}} + \mathfrak{P}_+ + \mathfrak{P}_-$ gives a realization of the Lie algebra $SO(2n, \mathbb{C})$.

So far, the dual representations of $Sp(N)$ and $SO^*(2n)$ behave exactly as the dual representations of $O(N)$ and $Sp(2n, \mathbb{R})$ as treated in section 3, but, as we shall see, there will be a quite remarkable and interesting difference when we restrict the action of $Sp(N)$ to a subgroup isomorphic $Sp(N-2)$. This difference arises not because there is no natural subgroup $Sp(N-1)$, so that the restriction goes down by two steps (as will be discussed in the conclusion, the treatment of the restriction of the actions of $U(N)$ to $U(N-M)$ or $O(N)$ to $O(N-M)$, $1 \leqslant M \leqslant N-1$, is a straightforward generalization of the case $M = 1$), but because of the action of the group $SO^*(2n)$ on the Heisenberg group $H_n^{N-1} \times H_n^N$, which we shall define shortly, is quite different from the action of the group $U(n)$ on $H_n^{N-1} \times H_n^N$, for example. This interesting phenomenon leads us to search for a group sitting between $Sp(N)$ and $Sp(N-2)$ which would play the role of the 'missing subgroup $Sp(N-1)$'; this will be investigated in section 5.

Choose $\sigma = \sigma_{N/2}$ of the form

$$\left(\begin{array}{c|c} \sigma_{N/2-1} & 0 \\ \hline 0 & J \end{array}\right)$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $\sigma_{N/2-1}$ is a $(N-2) \times (N-2)$ matrix such that $(\sigma_{N/2-1})^{-1} = -\sigma_{N/2-1}\sigma_{N/2-1}^{\mathrm{T}}$. Let $G \equiv G_N$ be defined by this form σ and identify the subgroup G_{N-2} defined by $\sigma_{N/2-1}$ with the subgroup of G which consists of all matrices of the form

$$\left(\begin{array}{c|c} \gamma & 0 \\ \hline 0 & 1_2 \end{array}\right) \qquad \gamma \in G_{N-2}.$$

If $\widetilde{R}^G$ denotes the restriction of R^G to G_{N-2}, then a system of generators of the infinitesimal action of $\widetilde{R}^G$ is given by

$$\widetilde{R}^G_{ab} = \sum_{\substack{\alpha=1,\ldots,n \\ k=1,\ldots,N-2}} \left(Z_{\alpha a}(\sigma_{N/2-1})_{kb} + Z_{\alpha b}(\sigma_{N/2-1})_{ka}\right) \frac{\partial}{\partial Z_{\alpha k}} \qquad 1 \leqslant a, b \leqslant N-2. \tag{4.4}$$

The infinitesimal dual action of $\widetilde{R}^G$ is a representation of the semidirect product $SO^*(2n) \oplus_{\mathrm{d}\tau} \mathcal{H}_{n,n}$ on $\mathcal{F}(\mathbb{C}^{n\times N})$. The action of the Heisenberg algebra on $\mathcal{F}(\mathbb{C}^{n\times N})$ is defined by the operators

$$Z_{\gamma,N-1} \qquad \frac{\partial}{\partial Z_{\gamma,N-1}} \qquad Z_{\gamma,N} \quad \frac{\partial}{\partial Z_{\gamma,N}} \qquad 1 \leqslant \gamma \leqslant n$$

which generate a direct sum of two Heisenberg algebras. However, the action of $SO^*(2n)$ on this direct sum is indecomposable as seen by the following equation for all $\alpha, \beta, \gamma = 1, \ldots, n$:

$$\begin{aligned}
&[P_{\alpha\beta}, Z_{\gamma,N-1}] = 0 \qquad [D_{\alpha\beta}, Z_{\gamma,N-1}] = \delta_{\alpha\gamma} \frac{\partial}{\partial Z_{\beta,N}} - \delta_{\beta\gamma} \frac{\partial}{\partial Z_{\alpha,N}} \\
&[E_{\alpha\beta}, Z_{\gamma,N-1}] = \delta_{\beta\gamma} Z_{\alpha,N-1} \\
&\left[P_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma,N-1}}\right] = \delta_{\alpha\gamma} Z_{\beta,N} - \delta_{\beta\gamma} Z_{\alpha,N} \\
&\left[D_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma,N-1}}\right] = 0 \qquad \left[E_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma,N-1}}\right] = -\delta_{\alpha\gamma} \frac{\partial}{\partial Z_{\beta,N-1}} \\
&[P_{\alpha\beta}, Z_{\gamma,N}] = 0 \qquad [D_{\alpha\beta}, Z_{\gamma,N}] = \delta_{\beta\gamma} \frac{\partial}{\partial Z_{\alpha,N-1}} - \delta_{\alpha\gamma} \frac{\partial}{\partial Z_{\beta,N-1}} \\
&[E_{\alpha\beta}, Z_{\gamma,N}] = \delta_{\beta\gamma} Z_{\alpha,N} \\
&\left[P_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma,N}}\right] = \delta_{\beta\gamma} Z_{\alpha,N-1} - \delta_{\alpha\gamma} Z_{\beta,N-1} \\
&\left[D_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma,N}}\right] = 0 \qquad \left[E_{\alpha\beta}, \frac{\partial}{\partial Z_{\gamma,N}}\right] = -\delta_{\alpha\gamma} \frac{\partial}{\partial Z_{\beta,N}}.
\end{aligned} \tag{4.5}$$

Let

$$\begin{aligned}
\widetilde{P}_{\alpha\beta} &= -\sum_{i,j=1,\ldots,N-2} Z_{\alpha j} (\sigma_{N/2-1})_{ji} Z_{\beta i} \\
\widetilde{D}_{\alpha\beta} &= \sum_{i,j=1,\ldots,N-2} \frac{\partial}{\partial Z_{\alpha j}} (\sigma_{N/2-1})_{ji} \frac{\partial}{\partial Z_{\beta,i}} \qquad 1 \leqslant \alpha, \beta \leqslant n \\
\widetilde{E}_{\alpha\beta} &= \sum_{i=1,\ldots,N-2} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}} + \frac{1}{2}(N-2)\delta_{\alpha\beta}.
\end{aligned} \tag{4.6}$$

Let $\widetilde{E}$, $\widetilde{P}$, $\widetilde{D}$, denote the matrices whose (α, β) entry, $1 \leqslant \alpha, \beta \leqslant n$, is $\widetilde{E}_{\alpha\beta}$, $\widetilde{P}_{\alpha\beta}$, and $\widetilde{D}_{\alpha\beta}$, respectively, as given by (4.6), and set

$$[\widetilde{L}^{G'}] = \begin{bmatrix} \widetilde{E} & \widetilde{P} \\ \widetilde{D} & -\widetilde{E}^{\mathrm{T}} \end{bmatrix}$$

then we have the following theorem.

Theorem 4. Let $H_n \times H_n$ denote the Heisenberg group whose infinitesimal action on $\mathcal{F}(\mathbb{C}^{n\times N})$ is given by the generators $\{Z_{\gamma,N-1}, \partial/(\partial Z_{\gamma,N-1}), Z_{\gamma,N}, \partial/(\partial Z_{\gamma,N}\}$, $\gamma = 1, \ldots, n$ then the algebra of all Casimir invariant differential operators of the representation of the semidirect product $SO^*(2n) \ltimes_\tau (H_n \times H_n)$ which is dual to the representation $\widetilde{R}^G$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ is generated by
(i) the constants and the n algebraically independent Casimir operators $\mathrm{Tr}([\widetilde{L}^{G'}]^{2s})$, $1 \leqslant s \leqslant n$, if $n < N/2$;
(ii) the constants and the $N/2$ algebraically independent Casimir operators $\mathrm{Tr}([L^{G'}]^{2s})$, $1 \leqslant s \leqslant N/2$, if $n \geqslant N/2$.

Proof. We have the decompositions

$$\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(\mu_{G_{N-2}})} \oplus\left(\mathcal{I}^{(\mu_{G_{N-2}})}(\mathbb{C}^{n\times(N-2)}) \otimes \mathcal{F}(\mathbb{C}^{n\times 2})\right) \tag{4.7a}$$

$$\mathcal{F}(\mathbb{C}^{n\times(N-2)}) = \sum_{(\mu_{G_{N-2}})} \oplus\mathcal{I}^{(\mu_{G_{N-2}})}(\mathbb{C}^{n\times(N-2)}) \tag{4.7b}$$

where in (4.7a) the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$ is decomposed into a direct sum of isotypic components under the dual actions of $G_{N-2} \times I_2$ and $SO^*(2n) \ltimes_\tau (H_n \times H_n)$ and in (4.7b) the Fock space $\mathcal{F}(\mathbb{C}^{n\times(N-2)})$ is decomposed under the dual actions of G_{N-2} and $G'_{N-2} \approx SO^*(2n, \mathbb{R})$. Thus if
(i) $n < N/2$ then $n \leqslant N/2 - 1$ and $(\mu_{G_{N-2}})$ corresponds to the signatures

$$\underbrace{(\ell_1, \ldots, \ell_n, 0, \ldots, 0)}_{N/2-1}$$

of G_{N-2} and $(\ell_1, \ldots, \ell_n)$ of G'_{N-2}; $\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant 0$.
(ii) $n \geqslant N/2$ then $n > N/2 - 1$ and $(\mu_{G_{N-2}})$ corresponds to the signatures $(\ell_1, \ldots, \ell_{N/2-1})$ of G_{N-2} and

$$\underbrace{(\ell_1, \ldots, \ell_{N/2-1}, 0, \ldots, 0)}_{n}$$

of G'_{N-2}.

As in section 3, it follows from [7] and theorem 1 that the algebra of all Casimir invariants of the action of $SO^*(2n) \ltimes_\tau (H_n \times H_n)$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ is generated by the system $\{\mathrm{Tr}([\widetilde{L}^G]^{2s})\}_s$, $1 \leqslant s \leqslant n$ if $n < N/2$, and $1 \leqslant s \leqslant N/2 - 1$ if $N \geqslant N/2$. □

5. Dual representations of the chains $Sp(N-2) \subset Sp(N-2) \oplus H_{N/2-1} \subset Sp(N)$ and $SO^*(2n) \oplus (\mathcal{H}_{n,n}) \supset SO^*(2n) \oplus \mathcal{A}_{n,n} \supset SO^*(2n)$

In this section we let $N = 2k$ and fix a symplectic form

$$\sigma \equiv \sigma_k = \left(\begin{array}{c|c} \sigma_{k-1} & 0 \\ \hline 0 & J \end{array}\right)$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_{k-1} = \begin{pmatrix} 0 & -1_{k-1} \\ 1_{k-1} & 0 \end{pmatrix}.$$

Then the Lie algebra $\mathcal{G}^{\mathbb{C}} = sp(2k, \mathbb{C})$ consists of all matrices of the form

$$\begin{array}{c} \\ k-1\{ \\ k-1\{ \\ 2\{ \end{array} \left(\begin{array}{c|c|c} \overbrace{X_1}^{k-1} & \overbrace{X_2}^{k-1} & \overbrace{W_1}^{2} \\ \hline X_3 & -X_1^{\mathrm{T}} & W_2 \\ \hline JW_2^{\mathrm{T}} & -JW_1^{\mathrm{T}} & V \end{array}\right)$$

where X_2 and X_3 are complex symmetric matrices and the complex 2×2 matrix V is such that $VJ = -JV^{\mathrm{T}}$. Recall that since M_{ij} denotes the $2k \times 2k$ matrix with 1 as the (i, j) entry and 0 elsewhere, we can choose a basis for $sp(2k, \mathbb{C})$ as follows:

$$\text{(I)} \qquad \begin{cases} f_{ij} = M_{ij} - M_{j+k-1,i+k-1} \\ g_{ij} = M_{i+k-1,j} + M_{j+k-1,i} & 1 \leqslant i, j \leqslant k-1 \\ h_{ij} = M_{i,j+k-1} + M_{j,+i+k-1} \end{cases}$$

$$\text{(II)} \qquad \begin{cases} f_{i,2k-1} = M_{i,2k-1} - M_{2k,i+k-1} \\ g_{i+k-1,2k-1} = M_{i+k-1,2k-1} + M_{2k,i} & 1 \leqslant i \leqslant k-1 \\ f_{i,2k} = M_{i,2k} + M_{2k-1,i+k-1} \\ h_{i+k-1,2k} = M_{i+k-1,2k} - M_{2k-1,i} \end{cases} \qquad (5.1)$$

$$\text{(III)} \qquad \begin{cases} f_{2k-1} = M_{2k-1,2k-1} - M_{2k,2k} \\ g_{2k-1} = M_{2k-1,2k} \\ h_{2k-1} = M_{2k,2k-1} \end{cases}$$

where in (5.1) (I) forms a basis for $sp(2k-2, \mathbb{C})$ and (III) forms a basis for $s\ell(2, \mathbb{C})$. Recall that

$$R_{ij} = \sum_{\alpha=1}^{n} Z_{\alpha i} \frac{\partial}{\partial Z_{\alpha j}} \qquad 1 \leqslant i, j \leqslant 2k$$

then the representation of $sp(2k, \mathbb{C})$ in $\mathcal{F}(\mathbb{C}^{n\times 2k})$ which maps M_{ij} onto R_{ij}, $F_{ij} \to R_{ij} - R_{j+k-1,i+k-1}$, etc, is faithful. By an abuse of language, let f_{ij}, g_{ij}, h_{ij}, etc, denote the images of f_{ij}, g_{ij}, h_{ij}, etc. under this representation.

Let $\mathcal{H}_{k-1}$ denote the Lie algebra generated by $f_{i,2k-1}$, $g_{i+k-1,2k-1}$, and h_{2k-1}, $1 \leqslant i \leqslant k-1$. Then it is easy to verify that

$$\begin{aligned} &[f_{i,2k-1}, f_{j,2k-1}] = 0 \\ &[g_{i+k-1,2k-1}, g_{j+k-1,2k-1}] = 0 \qquad 1 \leqslant i, j \leqslant k-1 \\ &[f_{i,2k-1}, g_{j+k-1,2k-1}] = \delta_{ij}(-2h_{2k-1}) \\ &[f_{i,2k-1}, h_{2k-1}] = 0 \qquad [g_{i+k-1,2k-1}, h_{2k-1}] = 0. \end{aligned} \qquad (5.2)$$

It follows immediately that $\mathcal{H}_{k-1}$ is a Heisenberg algebra of rank $k-1$. An easy computation also shows that

$$\begin{aligned} &[f_{ij}, f_{\ell,2k-1}] = \delta_{j\ell} f_{i,2k-1} \\ &[f_{ij}, g_{\ell+k-1,2k-1}] = -\delta_{i\ell} g_{j+k-1,2k-1} \\ &[g_{ij}, f_{\ell,2k-1}] = \delta_{j\ell} g_{i+k-1,2k-1} + \delta_{i\ell} g_{j+k-1,2k-1} \\ &[g_{ij}, g_{\ell+k-1,2k-1}] = 0 \qquad 1 \leqslant i, j, \ell \leqslant k-1 \\ &[h_{ij}, f_{\ell,2k-1}] = 0 \\ &[h_{ij}, g_{\ell+k-1,2k-1}] = \delta_{j\ell} f_{i,2k-1} + \delta_{i\ell} f_{j,2k-1} \\ &[f_{ij}, h_{2k-1}] = 0 \qquad [g_{ij}, h_{2k-1}] = 0 \qquad [h_{ij}, h_{2k-1}] = 0. \end{aligned} \qquad (5.3)$$

It follows that we have a semidirect sum $Sp(2k-2,\mathbb{C}) \oplus \mathcal{H}_{k-1}$ of Lie algebras. Similarly if we let $\mathcal{H}^{\surd}_{k-1}$ denote the Lie algebra generated by $f_{i,2k}$, $h_{i+k-1,2k}$, g_{2k-1}, $1 \leqslant i \leqslant k-1$, then $\mathcal{H}^{\surd}_{k-1}$ is a Heisenberg algebra isomorphic to $\mathcal{H}_{k-1}$. In fact the semidirect sums $Sp(2k-2,\mathbb{C})\oplus\mathcal{H}_{k-1}$ and $Sp(2k-2,\mathbb{C})\oplus\mathcal{H}^{\surd}_{k-1}$ are the two isomorphic maximal Lie algebras sitting between $Sp(2k-2,\mathbb{C})$ and $Sp(2k,\mathbb{C})$. It can be shown the dual representation of $Sp(2k-2,\mathbb{C}) \oplus \mathcal{H}_{k-1}$ ($Sp(2k-2,\mathbb{C}) \oplus \mathcal{H}^{\surd}_{k-1}$) on $\mathcal{F}(\mathbb{C}^{n\times 2k})$ is a representation of $SO^*(2n) \oplus \mathcal{A}_{n,n}$ ($SO^*(2n) \oplus \mathcal{A}^{\surd}_{n,n}$) on $\mathcal{F}(\mathbb{C}^{n\times 2k})$, where $\mathcal{A}_{n,n}$ ($\mathcal{A}^{\surd}_{n,n}$) is the Abelian Lie algebra defined by the generators

$$\begin{gathered}\left\{Z_{\gamma,2k},\ \frac{\partial}{\partial Z_{\alpha,2k-1}};\ 1\leqslant\alpha\leqslant n\right\}\\ \left(\left\{Z_{\alpha,2k-1},\ \frac{\partial}{\partial Z_{\alpha,2k}};\ 1\leqslant\alpha\leqslant n\right\}\right).\end{gathered} \tag{5.4}$$

It is also easy to verify that $SO^*(2n) \oplus \mathcal{A}_{n,n}$ ($SO^*(2n) \oplus \mathcal{A}^{\surd}_{n,n}$) is a semidirect product of a simple Lie algebra with an Abelian Lie algebra. However, neither the Casimir operators of $Sp(2k-2,\mathbb{C}) \oplus \mathcal{H}_{k-1}$ nor of $SO^*(2n) \oplus \mathcal{A}_{n,n}$ are easy to compute.

6. Conclusion

We have shown how to compute the Casimir operators of semidirect products of the groups $U(n)$ and $Sp(2n,\mathbb{R})$ with the n-dimensional Heisenberg group as well as the semidirect product of $SO^*(2n)$ with the direct sum of the two Heisenberg groups. In every case the procedure for computing all the (algebraically independent) Casimir operators was to associate a representation of the semidirect product with a dual representation of the compact groups $U(N-1)$, $O(N-1)$, and $Sp(N-2)$, respectively (for $Sp(N)$, N must be even) on the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$. Theorems 2, 3, and 4 prove that the semidirect product Casimir operators are equal to the algebraically independent Casimir operators of $U(N-1)$, $O(N-1)$, and $Sp(N-2)$, respectively, Casimir operators whose forms are well known.

Our procedure for obtaining Casimir operators is easily generalized to semidirect products of $U(n)$, $Sp(2n,\mathbb{R})$, and $SO^*(2n)$ with direct sums of Heisenberg groups. For the semidirect product of $U(n)$ with $H_n \oplus \cdots \oplus H_n$, where the direct sum is taken M times, the Casimir operators are just those of $U(N-M)$, $1 \leqslant M \leqslant N-1$. Similarly, the semidirect product of $Sp(2n,\mathbb{R})$, with $H_n \oplus \cdots \oplus H_n$ has Casimir operators equal to the Casimir operators of $O(N-M)$, $1 \leqslant M \leqslant N-2$. The semidirect product of $SO^*(2n)$ with a direct sum of Heisenberg groups is only defined for M even. The Casimir operators are then those of $Sp(N-M)$, N, M even, $2 \leqslant M \leqslant N-2$.

Besides the semidirect products discussed in this paper, there are other semidirect products of interest whose Casimir invariants can be obtained by the duality arguments used in this paper. For example, the semidirect product $U(n) \ltimes H_n$, discussed in section 2 has the subgroup $O(n) \ltimes H_n$, with a new set of Casimir operators. In this case $U(n) \ltimes H_n$ is dual to $U(N-1)$, so that $O(n) \ltimes H_n$ will be dual to a group containing $U(N-1)$, namely $Sp(2(N-1),\mathbb{R})$, whose Casimir operators are known. Casimir operators for groups like $O(n) \ltimes H_n$ and $Sp(n) \ltimes H_n$ (and direct sums of H_n) will be discussed in a subsequent paper.

It is well known that the Gel'fand–Žetlin scheme for labeling basis elements in representation spaces by chains of subgroups works for the unitary and orthogonal, but not for the symplectic groups. For the symplectic group chain, $Sp(N) \supset Sp(N-2) \supset$

$\cdots \supset Sp(2)$, N even, a given irreducible representation of $Sp(N)$ will contain irreducible representations of $Sp(N-2)$ *with multiplicity in general greater than one.*

By going to the complexification of the compact group $Sp(N)$, we have found a group 'between' $Sp(N, \mathbb{C})$ and $Sp(N-2, \mathbb{C})$, namely a semidirect product group $Sp(N-2, \mathbb{C}) \ltimes H_{N-2}$. In fact, there are two such semidirect products, as seen in section 5, equation (5.3) ff. Further we have shown that $Sp(N-2, \mathbb{C}) \ltimes H_{N-2}$ is a dual to $SO^*(2n) \ltimes \mathcal{A}_{n,n}$, the semidirect product of $SO^*(2n)$ with an Abelian group, with two different Abelian groups corresponding to the two different Heisenberg groups.

However, *it is not possible to use the theorems proved in this paper to compute the* Casimir operators of $Sp(N-2, \mathbb{C}) \ltimes H_{N-2}$. The problem of finding these Casimir operators and then using their eigenvalues to resolve the multiplicity of the symplectic group chain will be discussed in a subsequent paper.

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